Using vectors to understand a mathematical model of a neuron

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Introduction

The human brain is an information processing device. Information enters through the senses and the brain's output is behavior, mainly in the form of signals to muscles. Computation in a human brain is associated with the activity of a very large number neurons, estimated as 85×10^9 in (Herculano-Houzel, 2009).

The neural network in your brain is formed by connecting the neurons to $10,000 - 100,000$ other neurons in complex patterns - forming the unfathomable structure in your head, that currently decodes this text and controls your behavior. Both of basic interest and because we need to understand diseases caused by malfunctioning, we measure and model information processing in the human brain.

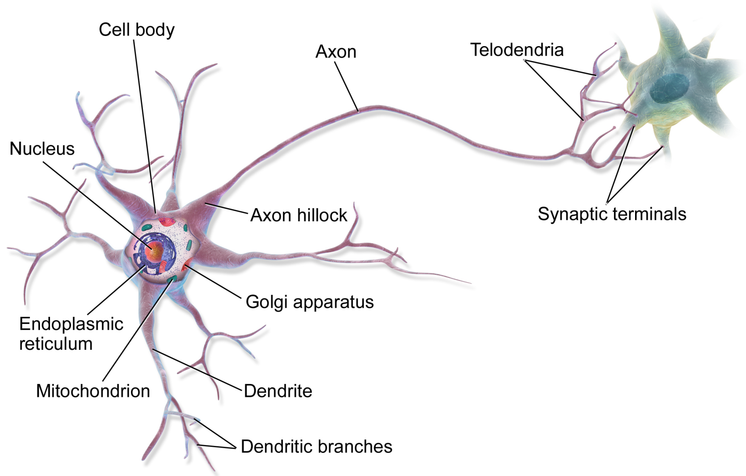

Figure 1: A schematic of the neuron: The 'computational unit in the brain' (Blausen Medical Communications). We model the neuron as a simple on-off computer. The neuron accepts input from other neurons and the senses through the dendritic tree. Based on the received input the neuron decides whether to be active or not. When becoming active the neuron communicates the change by signaling through the axon to downstream neurons before returning to the in-active state.

In this exercise we will analyse a simplified mathematical neuron model, for references see e.g., McCulloch and Pitts (1943); Widrow and Hoff (1960); Rosenblatt (1961). In our model the neuron is assumed to be in one of two states: active or in-active. Active neurons signal their state to receiving neurons downstream in the network. For a given neuron, the decision to become active is based on signals received from its input neurons through the dendritic tree, see figure 1. The effect of an incoming signal can be excitatory - contributing towards becoming active - or inhibitory, hence suppressing activity. The positive/negative modulation of the received input is thought to be learned from data and is implemented in the synaptic connection between an axon of the sending neuron and the dendritic tree of a receiving neuron.

In the mathematical model, the neuron computes a weighted sum of its D input

$$a = w_1 \cdot x_1 + w_2 \cdot x_2 + ... + w_D \cdot x_D, \tag{1}$$

The weights represent the effect of the synaptic connections and are positive (excitatory) or negative (inhibitory). The neuron's decision to become active is based on the value of a compared to a threshold t_0: The neuron is active if $a \geq t_0$ and in-active if $a < t_0$.

A two input vector model

To develop intuition on the decisions made by a neuron, let us first study a neuron with only two inputs x_1 and x_2,

$$a(\mathbf{x}) = w_1 \cdot x_1 + w_2 \cdot x_2 = \mathbf{w} \cdot \mathbf{x}. \tag{2}$$

Here we used that the computation of $a(\mathbf{x})$ can be understood as the 'dot product' between the two vectors $\mathbf{w} = (w_1, w_2)$ and $\mathbf{x} = (x_1, x_2)$. To characterize the decision process we consider the decision boundary, formed by vectors $\mathbf{x}$ that solve $a(\mathbf{x}) = t_0$ or

$$\mathbf{w} \cdot \mathbf{x} = t_0. \tag{3}$$

Assuming $w_2 \neq 0$ we can solve the equation for x_2 to get

$$x_2 = -\frac{w_1}{w_2} x_1 + \frac{t_0}{w_2}. \tag{4}$$

Hence the decision boundary formed by all the $\mathbf{x}$ vectors for which $a(\mathbf{x}) = t_0$, takes the form of a straight line with slope $\alpha = -\frac{w_1}{w_2}$ and intercept $\beta = \frac{t_0}{w_2}$. See Figure 2.

Exercise 1 Derive Equation (4) starting from Equation (3)

We can make a geometric interpretation of the dot product. Without loss of generality we will assume $|\mathbf{w}| = 1$ or $\sqrt{w_1^2 + w_2^2} = 1$. Rewrite the vectors as

$$\begin{aligned} \mathbf{w} &= (\cos v_w, \sin v_w) \\ \mathbf{x} &= |\mathbf{x}| (\cos v_x, \sin v_x) \end{aligned} \tag{5}$$

where v_w and v_x are angles between the respective vector and the first axis. We then get

$$a(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} = |\mathbf{x}| (\cos v_w \cos v_x + \sin v_w \sin v_x) = |\mathbf{x}| \cos(v_w - v_x). \tag{6}$$

Since, $v_w - v_x$ is the angle between the two vectors, we see that $a(\mathbf{x})$ is the projection of vector $\mathbf{x}$ on unit vector $\mathbf{w}$. See also Figure 2.

Exercise 2 Use Equation (6) to prove that $\mathbf{w}$ is the normal vector for the line where $a(\mathbf{x}) = t_0$. Hint: Show that the vector $\mathbf{x}' - \mathbf{x}$ is orthogonal to $\mathbf{w}$.

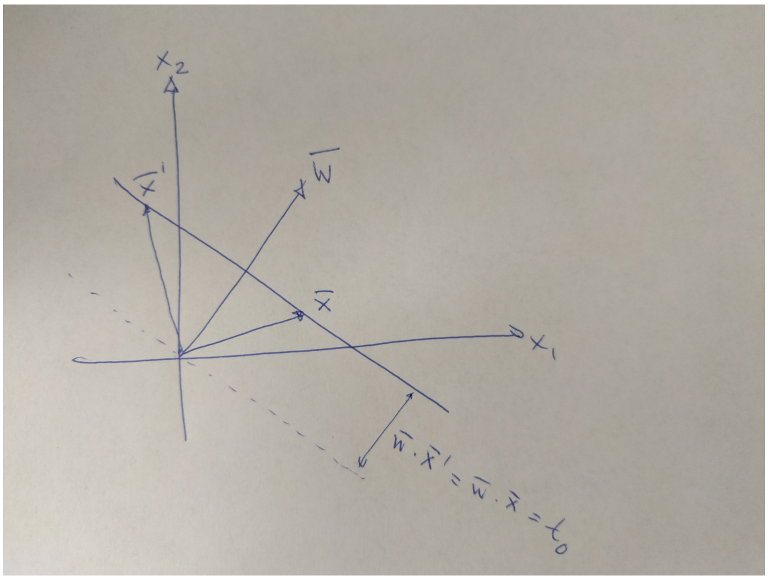

Figure 2: The $\mathbf{x}$ vectors solving $a(\mathbf{x}) = t_0$ form a line with slope $\alpha = -\frac{w_1}{w_2}$ and intercept with the second axis $\beta = \frac{t_0}{w_2}$.

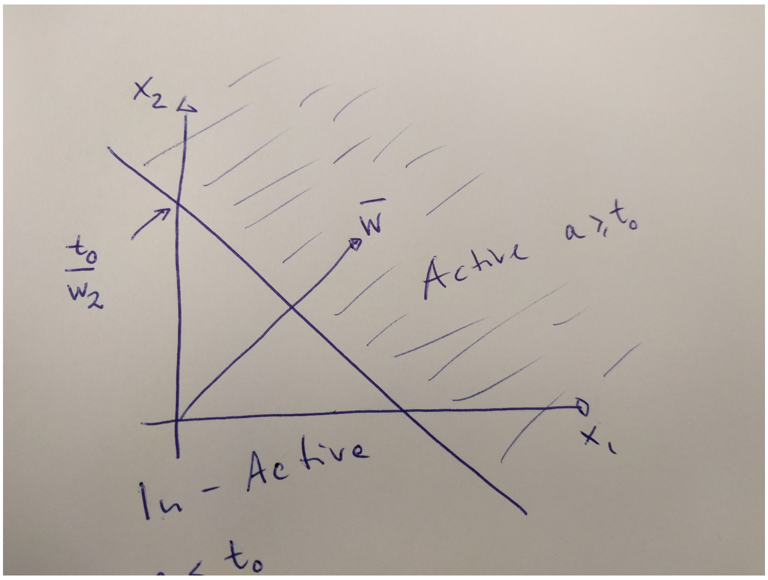

Figure 3: The line of $\mathbf{x}$s solving $a(\mathbf{x}) = t_0$ separates the two regions of inputs that lead to active and in-active neuron states.

The analysis shows that within our mathematical model the neuron serves as a basic pattern detector: The neuron becomes active for certain combinations of inputs that lead to $a(\mathbf{x}) \geq t_0$. Here, vectors of the input neurons' activity - the $\mathbf{x}$ vector - that are located on the positive side of the line, with respect to the normal vector $\mathbf{w}$.

Neurons in three dimensions and beyond

If we next consider a three dimensional case with inputs $\mathbf{x} = (x_1, x_2, x_3)$ and weights $\mathbf{w} = (w_1, w_2, w_3)$ we note that the two vectors $\mathbf{x}, \mathbf{w}$ of course can be viewed as vectors in a two dimensional plane within three dimensional space of possible inputs. Thus, the two-dimensional argument above can be now be carried out without changes, and $a(\mathbf{x})$ can be interpreted as the projection of $\mathbf{x}$ on unit vector $\mathbf{w}$.

What about the decision? Now $a(\mathbf{x}) = t_0$ is solved by a two-dimensional set of $\mathbf{x}$ vectors in the three dimensional space. To see this, assume that $w_3 \neq 0$. A simple generalization of our previous calculation shows

$$a(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} = t_0 \Rightarrow x_3 = -\frac{w_1}{w_3} x_1 - \frac{w_2}{w_3} x_2 + \frac{t_0}{w_3}. \tag{7}$$

The latter is the equation of a plane in 3D with the two slopes $\alpha_1 = -\frac{w_1}{w_3}$ and $\alpha_2 = -\frac{w_2}{w_3}$ and intercepting the third axis at $\beta = \frac{t_0}{w_3}$. Exercising a bit of faith we see this in fact holds for all

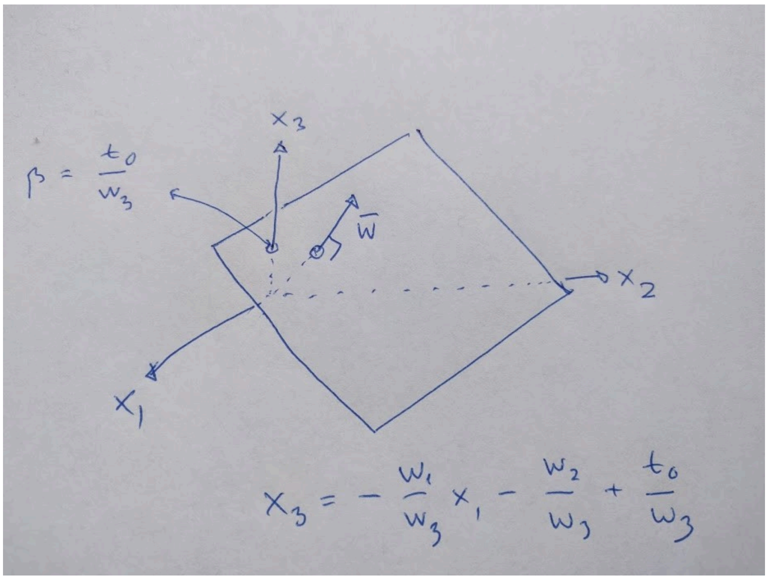

Figure 4: The decision surface of a neuron with three inputs.

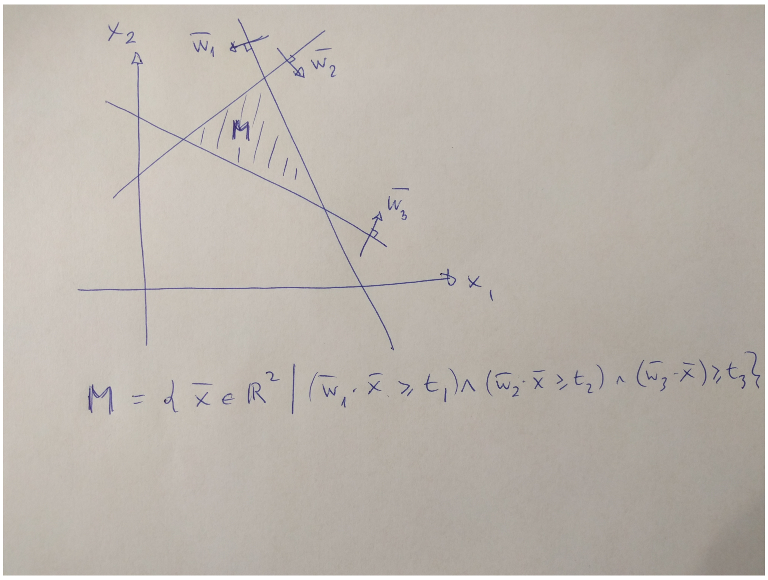

Figure 5: More complex decision regions can be formed by combining neurons.

dimensions, including the case of the human brain where the number of input neurons is $10,000 - 100,000$.

Exercise 4 Generalize the previous argument and show that the vector $\mathbf{w}$ is a normal vector to the two-dimensional plane of solutions to $a(\mathbf{x}) = t_0$ for a three dimensional $\mathbf{x}$.

More complex decisions can be implemented by neural networks

By combining neurons we can encode more complex decisions. Say we aim to detect input neuron states located in a triangular region in two dimensional input space. This can be implemented by combining three neurons as shown in Figure 5. Decisions are made by a fourth neuron accepting the activity of the three shown in Figure 5, as input, and having three weights $\mathbf{w}_{\text{output}} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and a threshold $t_{0,\text{output}} = \sqrt{3}$.

Exercise 5 Verify the above statement regarding weights and threshold of the 'output neuron'.

References

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